Differential equations

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1 Background

Recap from IX: Capacitors and Inductors

1.1 Ordinary differential equations

The fact that the V-I relationship of the capacitor and inductor contain time derivatives mean that to solve Kirchoff's laws for the current and voltage, we will now have to deal with currents and voltages that *change* with time, and we will want to solve *differential equations* for I and V to solve the circuits.

So, as a quick primer on differential equations, in case you have forgotten (if you are seeing this for the first time, I would recommend consulting a high school or first-year university textbook for slightly more background, differential equations are fun and important!):

If we have a linear homogeneous (meaning that the right-hand side is 0) differential equation (which all of our circuit equations will be):

$$f(t) + \alpha \frac{df}{dt} = 0$$

We can rearrange (treating the differentials as fractions):

$$\beta dt = -\alpha \frac{df}{f}$$

we can then integrate:

$$\int_0^t dt' = -\alpha \int_{f(0)}^{f(t)} \frac{df}{f}$$

soling the integral:

$$t = -\alpha \ln \frac{f(t)}{f(0)}$$

$$f(t) = f(0)e^{-t/\alpha}$$

Voilà! We have solved for $f(t)$, and just need to put in the *initial condition* $f(0)$ to find our exact function.

In hindsight, we could have also solved this differently, by simply making the *ansatz* that $f(t) = Ae^{\lambda t}$. Plugging that into our first equation, we get:

$$Ae^{\lambda t} + \alpha\lambda Ae^{\lambda t} = 0$$

since this has to hold true for all t, we can divide out the exponentials and A, and get what is called the *characteristic* or *auxilliary* equation, which is justan algebraic equation for λ, which we can solve:

$$\lambda = -\frac{1}{\alpha}$$

which means that we recover our solution: $f(t) = Ae^{-t/\alpha}$

Importantly, this kind of ansatz **also works for second-order differential equations** (see question 6).

Now, what do we do in the case where we have an *inhomogeneous* differential equation, such as this one?

$$f(t) + \alpha \frac{df}{dt} = \beta$$

It turns out this is easy to solve once we've already solved the corresponding homogeneous equation[1]! Namely, it can be shown that the general solution to the inhomogenous solution can be written as just the sum of *any* particular solution to the inhomogeneous equation and the solution to the corresponding homogenous equation. So, we just find a constant solution to the homogenous equation (verify that this holds)

$$f_{PI} = -\frac{\beta}{\alpha}$$

Now, our final inhomogeneous solution $f = f_{PI} + f_H$, where f_H is the solution to the homogeneous equation that we found above. So, our final solution to the inhomogeneous equation reads:

$$f = -\frac{\beta}{\alpha} + Ae^{-t/\alpha}$$

where A is a constant is to be determined by initial conditions.

Hence, we have a recipe for solving any linear differential equation:

1. Solve the corresponding honmogeneous equation, using your favorite technique (either rearrange treating the differentials as fractions, make an exponential ansatz, your favorite different technique!)
2. Find a particular solution to the inhomogeneous differential equation
3. Sum homogeneous and inhomogeneous solutions
4. Apply any initial conditions to determine remaining constants (there should be as many constants as the order of the equation, so 1 for first-order equations, 2 for second-order, etc).

2 Questions

1. Solve the following differential equations:
   (a) $y' = 0$
   (b) $y' = x$, $y(0) = y_0$
   (c) $y' = y$, $y(0) = y_0$
   (d) $y'' = 0$, $y'(0) = v_0$, $y(0) = y_0$
   (e) $\frac{dy}{dt} = t^2 + t + 1$, $y(0) = y_0$
   (f) $\frac{d^2y}{dt^2} = t^2 + t + 1$, $\frac{dy}{dt}(0) = v_0$, $y(0) = y_0$
2. Solve the following differential equations:
   (a) $y' = \frac{1}{y}$
   (b) $y' = \frac{1}{y^2}$
   (c) $y'' = y$, $y'(0) = 0$, $y(0) = y_0$
   (d) $y'' = -y$, $y'(0) = 0$, $y(0) = y_0$

[1] The proof of this is available in most first-year university courses on ordinary differential equations. See for instance `https://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/ODE/2018/ODELectureNotes.pdf`, §2.4.3 for more insights.

3. Solve the following differential equations
   (a) $y'' + 2y' + y = 0$
   (b) $y'' + 6y' + 9y = 4$, $y'(0) = 1$, $y(0) = 1$

4. A teapot with temperature 100°C in a room at temperature 23°C. According to Newton's formula, the rate of heat loss per unit area (in W/m^2) from the teapot is given by

$$q = h\Delta T$$

, where ΔT is the difference in temperature between the teapot and its surroundings.

   (a) Determine the units of h.
   (b) Show that the rate of change of temperature of the teapot is given by: $\frac{dT}{dt} = -\frac{hA(T-T_{room})}{C}$, where A is the surface area of the teapot, and C is the heat capacity of the teapot.
   (c) Solve for the temperature T of the teapot as a function of time. Find the characteristic cooling time τ of the teapot in terms of C, h, and A.
   (d) For a teapot with surface area $A = 0.06\text{m}^2$ $h = 12\ \text{Wm}^{-2}\text{K}^{-1}$, $C = 4.4 \times 10^3\ \text{JK}^{-1}$, find the characteristic time τ.

5. A penny of mass m is dropped from a height h. The penny experiences both gravitational force and a drag force from air resistance proportional to its velocity $F_d = -bv$, where b is a constant.
   (a) Write down the units of b.
   (b) Show that the equation of motion of the penny is given by:

$$m\ddot{y} = -mg - bv$$

   where y is the vertical position of the penny, and v is the velocity.
   (c) Find the terminal velocity v_{term} of the penny, in terms of m, g, and b.
   (d) Solve the equation of motion for the penny to find $y(t)$.
   (e) Sketch the velocity $v(t)$ of the penny, indicating the limit as time goes to inifinity.

6. A mass m is free to slide back and forth on a frictionless table, while attached to a spring of constant k and natural length x_0. At time $t = 0$, the mass has velocity v_0, and the spring is at its natural length.
   (a) Write down the equation of motion of the mass.
   (b) Solve the equation of motion for the position $x(t)$ of the mass as a function of time.
   (c) Sketch the motion of the mass $x(t)$. What is the period of oscillation?
   (d) Now, we add the consideration of air resistance. The mass now experiences a drag force $F_d = -\beta\dot{x}$, where β is a constant. Show that the equation f motion of the mass is now given by

$$m\ddot{x} = -kx - \beta\dot{x}$$

   (e) Solve the equation of motion for the spring, assuming that $\beta < 2\sqrt{km}$.
   (f) Sketch the solution you found for $x(t)$ in the case of air resistance. What happens in the case where $\beta > 2\sqrt{km}$?